# Math 409 Practice Final Exam

### This exam has 7 questions, for a total of 100 points.

Please answer each question in the space provided. No aids are permitted.

### Question 1. (20 pts)

For each of the following questions, circle the correct answer.

(a) 
$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - 1} =$$

- A.  $\frac{3}{2}$
- B.  $\infty$
- C.  $-\infty$
- D. 2
- (b) Let f and g be differentiable functions on  $\mathbb{R}$  such that  $f(0)=5,\ f'(0)=2,$  and g'(5)=3. Then  $(g\circ f)'(0)$  is equal to
  - A. 6
  - B. 5
  - C. 3
  - D. 2

(c) 
$$\lim_{x \to 0+} (1+2x)^{1/x} =$$

- A. 0
- B. 1
- C. 2
- D.  $e^2$

(d) 
$$\lim_{x \to \infty} \frac{\cos x}{x^2} =$$
A. 2
B. 1

$$D \propto$$

D. 
$$\infty$$

(e) Which of the following functions is not uniformly continuous on  $\mathbb{R}$ ?

A. 
$$f(x) = \frac{1}{x^2 + 1}$$
  
B.  $f(x) = 1 + x^2$ 

B. 
$$f(x) = 1 + x$$

C. 
$$f(x) = \sin x$$

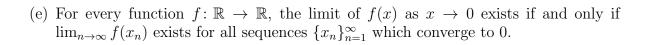
$$D. f(x) = \sin^2(x)$$

### Question 2. (24 pts)

In each of the following 8 cases, indicate whether the given statement is true or false. No justification is necessary.

- (a) The image of a continuous function  $f: \mathbb{R} \to \mathbb{R}$  is either finite or uncountable.
- (b) If  $\{x_n\}_{n=1}^{\infty}$  is a Cauchy sequence, then  $\{|x_n|\}_{n=1}^{\infty}$  is a Cauchy sequence.
- (c) If f is a bounded function on [0, 1], then there is an  $a \in [0, 1]$  such that f(a) = $\sup_{x \in [0,1]} f(x).$

	(4)	The	function	f(	~ \		<i>m</i> G	~	;	intorne	hla	0.70	$\Omega$	۲.	1
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(f) 
$$\{x_n\}_{n=1}^{\infty}$$
 is a sequence that converges and  $\{y_n\}_{n=1}^{\infty}$  is a sequence that does not converge, then the sequence  $\{x_ny_n\}_{n=1}^{\infty}$  dose not converge.

(g) If 
$$f:(0,1)\to\mathbb{R}$$
 is improperly integrable on  $(0,1)$ , then  $f^2$  is improperly integrable on  $(0,1)$ .

(h) Every nonempty subset of [0,1] has a supremum.

# Question 3. (12 pts)

(a) State the completeness axiom for the real numbers.

(b) State the Mean Value Theorem

(c) State the Intermediate Value Theorem

# Question 4. (12 pts)

Compute f' for each of the following functions  $f: \mathbb{R} \to \mathbb{R}$ .

(a) 
$$f(x) = e^{x^2}$$

(b) 
$$f(x) = \int_1^x \frac{t}{2 + \cos t} dt$$

(c) 
$$f(x) = \int_{1}^{x^2} e^{t^2} dt$$

# Question 5. (12 pts)

(a) State what it means for a function  $f: \mathbb{R} \to \mathbb{R}$  to be differentiable at a point  $a \in \mathbb{R}$ .

(b) Let f be a function on  $\mathbb{R}$  for which there exists a function g such that f(x) = xg(x) for all  $x \in \mathbb{R}$  and g is continuous at 0. Prove that f'(0) exists and determine its value.

Question 6. (10 pts) Define the function  $f: [0,2] \to \mathbb{R}$  by

$$f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 3, & x = 1 \\ 1, & 1 < x \le 2. \end{cases}$$

Prove directly from the definition of integrability that f is integrable on [0,2].

Question 7. (10 pts) Prove that the function  $f(x) = \frac{x}{\sqrt{x^6 + 1}}$  is improperly integrable on  $(0, \infty)$ .